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Some characterizations of egalitarian solutions on classes of TU-games

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Abstract

In this paper we derive characterizations of egalitarian solutions on two subclasses of the class of balanced games. Firstly we show that the Dutta–Ray solution is the only solution that satisfies symmetry, independence of irrelevant core allocations, and continuity on the class of convex games. Secondly, together with the other two requirements, a strengthening of continuity to monotonicity in the value of the grand coalition turns out to be sufficient for the characterization of the lexicographically maximal solution on the class of large core games.

Keywords: TU-game; Egalitarianism; Axiomatization

JEL classification: C71

1. Introduction

The notion of egalitarianism was first introduced in the context of TU-games by Dutta and Ray (1989). They showed that their interpretation of egalitarianism yields at most one egalitarian allocation for a given TU-game and that there is exactly one egalitarian allocation in case the game is convex. They also devised an algorithm to compute the egalitarian allocation for such convex games.

In the years after they published their paper two main branches started to develop in

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this research area. On one hand people were searching for egalitarian solutions that existed for larger classes of games then just convex games. Examples are Dutta and Ray (1991) who introduced a notion of constrained egalitarian allocation that existed for weakly superadditive games and Arin and Inarra (1997) who introduced several forms of egalitarian solutions based on the notion of the core and showed that these solutions exist for balanced games.

On the other hand there is the axiomatic approach. During the last few years several characterizations of the above-mentioned egalitarian solutions have been found, most of them by means of some form of consistency. For example Dutta (1990) showed that the egalitarian solution of Dutta and Ray (1989) is the only solution that is consistent both in the sense of Maschler and Davis and of Hart and Mas-Colell among all solutions that coincide with the egalitarian solution on two person games. Klijn et al. (1998) provide two characterizations of the Dutta–Ray solution using (weak versions of) these types of consistency separately. Both papers deal with the class of convex games. In Arin et al. (2000) the notion of independence of irrelevant core allocations is used to characterize egalitarian solutions as they are defined in Arin and Inarra (1997). Those results are stated on the domain of balanced games.

This paper consists in fact of two different parts. In the first part we consider the class of convex games. In Arin et al. (2000) it is already shown that any continuous core solution defined on the class of balanced games that satisfies symmetry and independence of irrelevant core allocations is (weakly) egalitarian. Specifically such a solution must coincide with the Dutta–Ray solution on the class of convex games. In this paper we will show that these axioms (continuity, symmetry and independence of irrelevant core allocations) characterize the Dutta–Ray solution on the class of convex games. This result is somewhat stronger than the result in Arin et al. (2000) in the sense that the restriction of the domain to convex games weakens the power of the IIC requirement.

The second part concerns large core games. These games were first introduced by Sharkey (1982). A game is said to have a large core if any (non-efficient) allocation that satisfies the core inequalities (i.e. it is an element of the upper core) dominates a core element of the game. These games arise naturally from economic situations involving cost allocation. We show that on this class of games an extension of the Dutta–Ray solution called the lexmax (lexicographically maximal) solution is axiomatized by symmetry, independence of irrelevant core allocations and monotony of the solution w.r.t. the value of the grand coalition. Moreover we show that a generalization of the algorithm designed by Dutta and Ray can be used to compute the lexmax solution on the class of large core games.

2. Preliminaries

A transferable utility game (or TU-game) v is a function $v\colon 2^N \to \mathbb{R}$ on the collection 2^N of subsets of a finite set N (the players of the game) such that $v(\emptyset) = 0$. Any subset S of the player set N is called a *coalition* and $v(S)$ is called the *worth* of coalition S.

A vector $x \in \mathbb{R}^N$ is called an *allocation* to the players. The ith coordinate x_i of the allocation x represents the payoff to player $i \in N$. Thus, the amount of money coalition S gets according to this allocation is equal to $x(S) := \sum_{i \in S} x_i$. The allocation x is called *efficient* if $x(N)$ equals the worth $v(N)$ of the grand coalition N. The *core* $C(v)$ of the game v is the set of efficient allocations x in $\mathbb{R}^N$ with

$$x(S) \geq v(S) \quad \text{for all } S \subset N.$$

A game v with a non-empty core is called *balanced*. Furthermore, a game v is called convex if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for all coalitions S and T in N. It is well-known that convex games are balanced.

Let $\mathscr{C}$ be a subset of the class of balanced games. A *solution* on $\mathscr{C}$ is a function that assigns to each game in $\mathscr{C}$ one single allocation of that game. It is called a *core solution* if for each game the allocation assigned to this game is a core element of the game.

2.1. Egalitarian solutions

Let x be an allocation in $\mathbb{R}^n$. By $\hat{x}$ we denote the vector that results from x by permuting the coordinates of x in such a way that $\hat{x}_1 \geq \hat{x}_2 \geq \cdots \geq \hat{x}_n$. We say that the vector x *Lorenz dominates* the vector y if

$$\sum_{i=1}^{k} \hat{x}_i \leq \sum_{i=1}^{k} \hat{y}_i \quad \text{for all } k \in \{1, \ldots, n\}$$

and if at least one of these inequalities is strict. In words: for every k the group of k richest people with respect to x must at most be as rich as the group of k richest people with respect to y (and there must be one strict inequality). A core allocation x is called *egalitarian* if no other core allocation Lorenz dominates x.

Observe that if for a game v the equal share payoff of $v(N)/|N|$ to the players constitutes a core allocation it is necessarily the only egalitarian allocation in this game.

A solution on a class $\mathscr{C}$ of balanced games is called *egalitarian* if it assigns to each game in the class an egalitarian allocation of that game.

We will now briefly discuss the egalitarian solutions we will encounter in this paper. First, we will discuss the lexicographically maximal and minimal solutions.

We say that an allocation x in $\mathbb{R}^N$ is lexicographically preferred to another allocation y in $\mathbb{R}^N$ if the first non-zero coordinate of the vector $\hat{y} - \hat{x}$ is positive. This is denoted by $x \succ_{\text{lex}} y$. By $x \succeq_{\text{lex}} y$ we mean $\hat{x} = \hat{y}$ or $x \succ_{\text{lex}} y$.

Now, for a balanced game v, we define the set Lmax(v) of lexicographically most preferred core allocations as the set of allocations x in $C(v)$ with $x \succeq_{\text{lex}} y$ for all y in $C(v)$. Analogously we define Lmin(v) as the set of allocations x in $C(v)$ with $-x \succeq_{\text{lex}} -y$ for all y in $C(v)$.

From Lemma 1.1 of Moulin (1988) it follows directly that both Lmax(v) and Lmin(v) are single-valued. Hence we get the following well-known result.

Theorem 1. Lmax *and* Lmin *are core solutions on the class of balanced games.*

It is straightforward to check that Lmax and Lmin are even egalitarian. In Section 3 we will focus our attention on the Lmax solution on the class of large core games.

For convex games the situation is fairly simple. It was shown by Arin and Inarra (1997) that a convex game admits precisely one egalitarian allocation. This allows us to speak of *the* egalitarian solution on the class of convex games. Moreover they showed that, for convex games, the solution introduced by Dutta and Ray (1989) coincides with this solution. For that reason we will refer to the unique egalitarian solution on the class of convex games as the Dutta–Ray solution.

2.2. Properties

In the axiomatizations of the egalitarian solutions presented above we will basically use symmetry and continuity of the solution together with a form of the axiom of independence of irrelevant alternatives.

Let us first pay some attention to the independence of irrelevant alternatives axiom. This property is widely used in the context of bargaining situations. In bargaining models it is the objective of agents to reach a 'best' agreement point given a non-empty set of possible agreement points. Informally, in bargaining situations the IIA axiom says the following. Suppose we have two bargaining situations, such that the set of possible agreement points of one problem is contained in the set of possible agreement points of the other problem. Furthermore, suppose that the solution of the larger problem is available in the smaller problem. Then the solution of the larger problem should also be the solution of the smaller problem. The interpretation is that the solution was already the 'best' point of the larger set. Hence, if no new alternatives are offered, and only irrelevant alternatives are canceled, then the solution should remain the same.

The use of IIA in the context of TU-games is less obvious than in the context of bargaining situations because changing a TU-game means changing the worth of some coalitions, and players may have the feeling that their 'strength' has either increased or decreased. Most solution concepts on TU-games reflect this notion of strength, and do therefore not satisfy such a condition. The nucleolus and the Shapley value for example do not.

However, egalitarian considerations originate from the idea that players have to negotiate about which allocation to choose *before they know which role in society they will occupy*, i.e. before they know which player they will be. This type of negotiation is usually referred to as negotiation behind the veil of ignorance. Under this assumption it is not so strange that, once the players have decided which allocation is 'best for society', they will stick to this decision when the number of possibilities is decreased. Possible allocations in this setting are identified with core allocations. Thus the core restrictions function as minimum requirements for an allocation to be acceptable to society. These ideas are formalized in

(i) (IIC). A core solution ϕ on a class $\mathscr{C}$ of balanced games is *independent of irrelevant core allocations* if $\phi(v)=\phi(w)$ for any two games v and w in $\mathscr{C}$ with $\phi(v)\in C(w)\subset C(v)$.

The other two axioms are fairly straightforward and widely accepted, so we will simply state them without further discussion.

(ii) Symmetry (SYM). A solution ϕ on $\mathscr{C}$ is said to be symmetric if for any game v in $\mathscr{C}$ it holds that $\phi_i(v)=\phi_j(v)$ whenever $v(S\cup\{i\})=v(S\cup\{j\})$ for all $S\subset N\backslash\{i, j\}$.

iii) Continuity (CON). A solution ϕ on $\mathscr{C}$ is continuous if $\lim_{n\to\infty} \phi(v^n)=\phi(v)$ whenever $\lim_{n\to\infty} v^n = v$ (provided of course that all games v^n and v are in $\mathscr{C}$.)

3. The egalitarian solution for convex games

One of the immediate consequences of Theorem 4 in Arin et al. (2000) is that any core solution on the class of balanced games that satisfies IIC, SYM and CON coincides with the Dutta–Ray solution on the class of convex games. In this section we will even show that any solution defined *only* on the class of convex games must be equal to the solution of Dutta and Ray as soon as it satisfies IIC, SYM and CON. Note that this is indeed a stronger result, since the restriction of the IIC property to the class of convex games yields a weaker requirement in comparison to this property taken on the class of all balanced games.

First we need to introduce some terminology. A *chain* is a collection $\mathscr{S}$ of coalitions such that for any two elements S and T of this collection at least one of the two inclusions $S\subset T$ and $T\subset S$ holds, and moreover $\emptyset$ and N are elements of $\mathscr{S}$. (These last two conditions are usually not a part of the definition of a chain, but in this paper it is convenient to do so.) Usually we will index the elements of $\mathscr{S}$ in such a way that $\emptyset=S_1\subset S_2\subset\cdots\subset S_k=N$.

Now let v be a convex game and let $\mathscr{S}$ be a chain. Then the *partial marginal* $p(v, \mathscr{S})\in\mathbb{R}^N$ is defined by

$$p(v, \mathscr{S})_i := \frac{v(S_{m+1})-v(S_m)}{|S_{m+1}\backslash S_m|}$$

where m is the unique index such that i is an element of $S_{m+1}\backslash S_m$.

Remark. In general a partial marginal need not be an element of the core of v. However, since v is convex, marginal vectors are examples of partial marginals that happen to be elements of the core of v.

Now let ϕ be a core solution, defined on the class of convex games, that satisfies IIC, SYM and CON. A crucial step in the proof of the main result of this section, that ϕ must be equal to the Dutta–Ray solution, is the observation stated below that $\phi(v)$ is a partial marginal of v. It is however an arduous task to prove it. Therefore we have put the results involved in this proof in Appendix A.

Theorem 2. *Let v be a convex game. Then $\phi(v)$ is a partial marginal of v.*

So, we want to show now that $\phi(v)$ equals the Dutta–Ray solution. First we will show this for games whose value of grand coalition is sufficiently large in comparison with the worths of the other coalitions.

Lemma 3. *Let w be a convex game such that $|S|w(N) \geq |N|w(S)$ for all coalitions S. Then $\phi(w)$ is equal to the Dutta–Ray allocation.*

Proof. First notice that, since $|S|w(N) \geq |N|w(S)$ for all coalitions S, the equal share allocation that gives $w(N)/|N|$ to each player i in N is a core allocation of w. Thus this is the only egalitarian allocation of the game w and therefore necessarily equal to the Dutta–Ray allocation. So, it suffices to show that $\phi(w)$ is equal to the equal share allocation.

In order to do that, choose

$$K := \min\left\{ \frac{w(S)}{|S|} \,\middle|\, \emptyset \neq S \subset N \right\}$$

and define the game u by

$$u(S) := \begin{cases} K|S| & \text{if } S \neq N \\ w(N) & \text{if } S = N. \end{cases}$$

It is straightforward to check that this game is convex. Now, since all players in u are symmetric, we know by SYM that

$$\phi(u)_i = \frac{u(N)}{|N|} = \frac{w(N)}{|N|}$$

for all players $i \in N$. Second, since for all non-empty $S \neq N$

$$u(S) = K|S| \leq \frac{w(S)}{|S|}\,|S| = w(S)$$

and $u(N) = w(N)$, we know that $C(w)$ is a subset of $C(u)$. Third, we already saw that the equal share allocation $\phi(u)$ is an element of the core of w. Hence, by IIC, $\phi(w)$ must be equal to the equal share allocation $\phi(u)$. □

In order to complete the proof of the main result of this section we need some insight in the behavior of core solutions that exclusively assign partial marginals. We use the following lemma, showing that, whenever two partial marginals in the core happen to coincide for a certain game, they either remain coincidental or at most one of them stays in the core when the value of the grand coalition is decreased.

Lemma 4. *Let w be a convex game. Suppose that there are two chains $\mathcal{S} = \{S_1, \ldots, S_k\}$ and $\mathcal{T} = \{T_1, \ldots, T_r\}$ such that $p(w, \mathcal{S}) = p(w, \mathcal{T})$. Further suppose that u is a convex game with $u(S) = w(S)$ for all coalitions $S \neq N$ and $u(N) < w(N)$. Then either $S_{k-1} = T_{r-1}$ and then automatically $p(u, \mathcal{S}) = p(u, \mathcal{T})$, or $S_{k-1} \neq T_{r-1}$ and then at least one of the two allocations $p(u, \mathcal{S})$ and $p(u, \mathcal{T})$ is not an element of the core of u.*

Proof. From the definition of a partial marginal it is easy to deduce that $p(u, \mathcal{S}) = p(u, \mathcal{T})$ if $S_{k-1} = T_{r-1}$.

So assume that S_{k-1} is not equal to T_{r-1}. By symmetry we can assume w.l.o.g. that there is a player j in $S_{k-1} \backslash T_{r-1}$. Now, since $p(w, \mathcal{S})$ is equal to $p(w, \mathcal{T})$, we know that

$$\sum_{i \in S_{k-1}} p(w, \mathcal{T})_i = \sum_{i \in S_{k-1}} p(w, \mathcal{S})_i = w(S_{k-1}).$$

However, player j is a member of the coalition $N \backslash T_{r-1}$. So, player j gets strictly less in $p(u, \mathcal{T})$ than in $p(w, \mathcal{T})$, while none of the players gets more in the first allocation than in the second one. Hence,

$$\sum_{i \in S_{k-1}} p(u, \mathcal{T})_i < \sum_{i \in S_{k-1}} p(w, \mathcal{T})_i = w(S_{k-1}) = v(S_{k-1}) = u(S_{k-1})$$

and $p(u, \mathcal{T})$ violates at least one core condition for the game u. □

Now consider the following setting. Let v be a convex game. Let $U(v)$ denote the collection of games w for which $w(S) = v(S)$ for all coalitions S not equal to N and $w(N) \geq v(N)$. Since v is convex it can easily be seen that all games in $U(v)$ are also convex.

Notice that $U(v)$ can be parameterized by the set of real numbers $\lambda \geq v(N)$ in the following way. For each $\lambda \geq v(N)$ the game $v(\lambda)$ in $U(v)$ is defined by

$$v(\lambda)(S) = \begin{cases} v(S) & \text{if } S \neq N \\ \lambda & \text{if } S = N. \end{cases}$$

In other words, the set $U(v)$ is a halfline in the cone of convex games parameterized by the value of the grand coalition.

Now consider two continuous functions ψ_1 and ψ_2 from $U(v)$ to $\mathbb{R}^N$ for which $\psi_1(w)$ and $\psi_2(w)$ are both partial marginals in the core of w for all games w in $U(v)$. We get the following result.

Lemma 5. *Suppose that $\psi_1(w)$ equals $\psi_2(w)$ for some w in $U(v)$. Then $\psi_1(v)$ equals $\psi_2(v)$.*

Proof. We will prove the above statement by contradiction. So, suppose that $\psi_1(v)$ is not equal to $\psi_2(v)$. Consider the set

$$\Lambda := \{\lambda \geq v(N) | \psi_1(v(\lambda)) = \psi_2(v(\lambda))\}.$$

Since $\psi_1(w)$ equals $\psi_2(w)$ for some w in $U(v)$ we know that Λ is not empty. So, since it is bounded from below, $\zeta := \inf(\Lambda)$ exists. Moreover, since $\psi_1(v)$ does not equal $\psi_2(v)$ and ψ_1 and ψ_2 are continuous, $\zeta > v(N)$. Thus, we can take an increasing sequence

$$\lambda_1, \lambda_2, \ldots$$

with $\lambda_k \geq v(N)$ that converges to ζ. Furthermore, since the collection of chains is finite we can, by switching to a subsequence if necessary, make sure that there are two chains $\mathscr{S}$ and $\mathscr{T}$ such that

$$\psi_1(v(\lambda_k)) = p(v(\lambda_k), \mathscr{S}) \quad \text{and} \quad \psi_2(v(\lambda_k)) = p(v(\lambda_k), \mathscr{T})$$

for all $k \in \mathbb{N}$. Thus, from the definition of ζ and the continuity of ψ_1, ψ_2 and the partial marginals we get that

$$p(v(\zeta), \mathscr{S}) = \psi_1(v(\zeta)) = \psi_2(v(\zeta)) = p(v(\zeta), \mathscr{T}).$$

Hence,

$$p(v(\zeta), \mathscr{S}) = p(v(\zeta), \mathscr{T})$$

and

$$p(v(\lambda_k), \mathscr{S}) \neq p(v(\lambda_k), \mathscr{T})$$

for all k. This however, contradicts Lemma 4 since by assumption all the above allocations must be core elements of the games in question. □

Now we can prove

Theorem 6. *The Dutta–Ray solution is the only core solution on the class of convex games that satisfies IIC, SYM and CON.*

Proof. From the results of Arin et al. (2000) it is clear that the Dutta–Ray solution of a convex game v is the unique core allocation allocation x in v for which

$$\|x\|_2 \leq \|y\|_2$$

holds for all core allocations y of the game v. (As usual, $\|\cdot\|_2$ denotes the Euclidean norm.) From this observation it immediately follows that the Dutta–Ray solution satisfies IIC and SYM. The continuity follows with a little bit more work from the same observation and the maximum theorem of Berge (1966).

So, we only have to show the converse statement. To this end, let ϕ be a core solution that satisfies IIC, SYM and CON. In order to show that ϕ is equal to the Dutta–Ray solution, let v be a convex game. We will show that $\phi(v)$ is equal to the Dutta–Ray solution of v.

First we will show that there is a game $v(\lambda^*)$ in $U(v)$ for which $\phi(v(\lambda^*))$ is equal to the Dutta–Ray solution of $v(\lambda^*)$. To this end, take

$$\lambda^* := \max\left\{ \frac{|N|v(S)}{|S|} \,\middle|\, \emptyset \neq S \subset N \right\}.$$

Then obviously $v(\lambda^*)$ satisfies the conditions of Lemma 3. So, $\phi(v(\lambda^*))$ equals the Dutta–Ray solution of the game $v(\lambda^*)$. Hence, since both ϕ and the Dutta–Ray solution are continuous (ϕ by assumption, and Dutta–Ray simply because it is continuous), and both solutions are core solutions that exclusively assign partial marginals (by Theorem 2 for both solutions), $\phi(v)$ equals the Dutta–Ray solution by Lemma 5. □

4. Egalitarianism for large core games

In this section we will give a characterization of the Lmax solution on the class of large core games. The three axioms used are IIC, SYM and AMON, a condition that (in combination with the other two) is somewhat stronger than CON. We will also provide a simple procedure for the calculation of Lmax. A similar procedure is already described in Arin and Inarra (1997) for the calculation of Lmin on the class of veto-rich games. Just like the procedure of Arin and Iñarra, our procedure reduces to the algorithm of Dutta and Ray (1989) when applied to the class of convex games.

Let v be a balanced game. As in the previous section we define $U(v)$ as the set of games w with $w(S)=v(S)$ for all $S \neq N$ and $w(N) \geq v(N)$.

The game v is said to have a *large core* if for all $w \in U(v)$ and for all $x \in C(w)$ there exists an allocation y in $C(v)$ such that $y \leq x$.

The procedure. We will now present a procedure that has a (large core) game v as input and produces a vector $\phi(v)$ as output.

Step 1. Let $\eta_1 = \eta_1(v) \in \mathbb{R}$ be the minimal number η for which

$$|S|\eta \geq v(S) \quad \text{for all } S \subset N.$$

Let $P_1(v)$ be the coalition of players $i \in N$ for which

$$|S|\eta_1 = v(S) \quad \text{for some } S \subset N \text{ with } i \in S.$$

Define $\phi(v)_i := \eta_1$ for all $i \in P_1(v)$.

Step k. Suppose that $P_{k-1}(v)$ is already defined and that $\phi(v)_j$ is already defined for all players j in $P_{k-1}(v)$. Let $\eta_k = \eta_k(v) \in \mathbb{R}$ be the minimal number η for which

$$\sum_{j \in P_{k-1}(v) \cap S} \phi(v)_j + |S \backslash P_{k-1}(v)|\eta \geq v(S) \quad \text{for all } S \subset N.$$

Let $P_k(v)$ be the coalition of players $i \in N$ for which

$$\sum_{j \in P_{k-1}(v) \cap S} \phi(v)_j + |S \backslash P_{k-1}(v)|\eta_k = v(S) \quad \text{for some } S \subset N \text{ with } i \in S.$$

Define $\phi(v)_i := \eta_k$ for all $i \in P_k(v) \backslash P_{k-1}(v)$.

Remarks. First notice that the coalition $P_k(v)$ strictly includes $P_{k-1}(v)$ as long as $P_{k-1}(v)$ is not the grand coalition N. Therefore it is clear that $P_{|N|}(v) = N$. So, $\phi(v)$ is defined for all players in at most $|N|$ steps, and the procedure can then be terminated.

One can think of this procedure as follows. In the first step we check for which games $v(\eta)$ in $U(v)$ with $v(\eta)(N) = \eta|N|$ the equal share solution $(\eta, \ldots, \eta)$ is an element of the core of the game $v(\eta)$. Once we have established the minimal value η_1 for which this is the case, we define $P_1(v)$ as the set of players that are a member of some coalition S that is tight[1] on the allocation $(\eta_1, \ldots, \eta_1)$ in the game $v(\eta_1)$. We now allocate η_1 to these players.

In the second step we check for which games $v(\eta)$ the allocation where the players in $P_1(v)$ get η_1 and the players outside $P_1(v)$ get η is still an element of the core of $v(\eta)$. The minimal value η for which this is the case is called η_2. Etcetera.

We will first try to show that the resulting vector $\phi(v)$ is in fact equal to the Lmax(v). To this end we need

Lemma 7. *The vector $\phi(v)$ is an element of $C(v)$.*

Proof. It is clear that

$$\phi(v)(S) \geq v(S) \quad \text{for all } S \subseteq N$$

by construction. So we only need to show that $\phi(v)(N) \leq v(N)$.

Define the game w by

$$w(S) := \begin{cases} v(S) & \text{if } S \neq N \\ \phi(v)(N) & \text{if } S = N \end{cases}$$

Then w is clearly an element of $U(v)$. It is also obvious that $\phi(v)$ is an element of $C(w)$. Moreover, since v has a large core, there must be an allocation y in the core of v with $y \leq \phi(v)$. Let j be an arbitrary player in N. We trivially have $y_j \leq \phi(v)_j$. We will also show that $y_j \geq \phi(v)_j$. By the construction of $\phi(v)$, there must be a coalition T with $j \in T$ and $\phi(v)(T) = v(T)$. Since $y \in C(v)$, we must also have $y(T) = v(T)$. Now, using that $y(T \backslash j) \leq \phi(v)(T \backslash j)$, we get that

$$y_j = y(T) - y(T \backslash j) = v(T) - y(T \backslash j) \geq v(T) - \phi(v)(T \backslash j) = \phi(v)_j.$$

This shows that $y_j = \phi(v)_j$ for an arbitrary $j \in N$, hence $y = \phi(v)$ is an element of the core of v. □

So now we know that ϕ is indeed a core solution on the class of large core games. The next Theorem identifies this solution with the Lmax solution.

Theorem 8. *The core solution ϕ equals* Lmax.

Proof. Let v be a large core game. We will show the equality of $\phi(v)$ and Lmax(v) by induction to the number of steps in the calculation of $\phi(v)$.

A. Consider $P_1(v)$. Take a player $j \in P_1(v)$ and a coalition S with $j \in S$ and

$$\phi(v)(S) = v(S).$$

Note that

[1] A coalition S is said to be *tight* on an allocation x in a game v if $x(S) = v(S)$.

$$\mathrm{Lmax}(v)_j \leq \widehat{\mathrm{Lmax}(v)}_1 \leq \widehat{\phi(v)}_1 = \phi(v)_j.$$

This inequality holds for all $i \in S$, and not one of these inequalities can be strict, since this would imply

$$\mathrm{Lmax}(v)(S) < \phi(v)(S) = v(S),$$

contradicting the fact that Lmax is a core solution. Hence, $\phi(v)_i = \mathrm{Lmax}(v)_i$ for all players $i \in P_1(v)$.

B. Suppose we already know that

$$\phi(v)_i = \mathrm{Lmax}(v)_i$$

for all players $i \in P_k(v)$. Take a player $j \in P_{k+1}(v) \backslash P_k(v)$ and a coalition S with $j \in S$ and

$$\phi(v)(S) = v(S).$$

By the construction of $\phi(v)$ we have $\phi(v)_j < \phi(v)_i$ for all $i \in P_k(v)$. Hence, by the induction hypothesis, $\phi(v)_j < \phi(v)_i = \mathrm{Lmax}(v)_i$ for all $i \in P_k(v)$. Then we must also have $\mathrm{Lmax}(v)_j < \mathrm{Lmax}(v)_i$ for all $i \in P_k(v)$. Otherwise we would get that $\phi(v) \succ_{lex} \mathrm{Lmax}(v)$, which would be in contradiction with the fact shown in Lemma 7 that $\phi(v)$ is a core element and the definition of $\mathrm{Lmax}(v)$. Hence, denoting $t = |P_k(v)| + 1$, it follows that

$$\mathrm{Lmax}(v)_j \leq \widehat{\mathrm{Lmax}(v)}_t \leq \widehat{\phi(v)}_t = \phi(v)_j.$$

This inequality holds for all $i \in S$, and not one of these inequalities can be strict, since this would imply

$$\mathrm{Lmax}(v)(S) < \phi(v)(S) = v(S),$$

contradicting the fact that Lmax is a core solution. Hence, $\phi(v)_i = \mathrm{Lmax}(v)_i$ for all players $i \in P_{k+1}(v)$ $\square$.

Using this result we will proceed with a characterization of the solution Lmax on the class of large core games. We already know that Lmax satisfies IIC, SYM and CON, even on the class of balanced games. These three axioms are not enough though to characterize Lmax on the class of large core games.

Example 5. Consider the four player game v defined by

$$v(S) := \begin{cases} 7 & \text{if } S = \{1, 2\} \\ 7 & \text{if } S = \{1, 3\} \\ 12 & \text{if } S = \{1, 2, 3, 4\} \\ 0 & \text{if } S = \{1\}, \{2\} \text{ or } \{3\} \\ -3 & \text{if } S = \{4\} \\ -4 & \text{else.} \end{cases}$$

First we will show that this game has a large core. To this end, let x be an allocation with $x(S) \geq v(S)$ for all $S \subset N$. We have to show that there exists an allocation y in $C(v)$

with $y \leq x$. Suppose that this is not the case. Then we may assume w.l.o.g. that $x(N) > 12$ and no coordinate of x can be decreased without violation of one of the inequalities $x(S) \geq v(S)$ for some $S \neq N$. Then $x_4 = -3$.

(a) If $x_2 = v(2) = 0$. Then $x_1 + x_3 \leq v(13) = 7$ so $x(N) \leq 0 + 7 - 3 < 12$ which contradicts $x(N) > 12$. Similarly, $x_3 = v(3)$ cannot be the case.
(b) From (a) it follows now that $x_1 + x_2 = 7$ and $x_1 + x_3 = 7$. Now we get $x(N) \leq 7 + 7 - 3 < 12$. Contradiction.

However, $\text{Lmax}(v) = (3\frac{1}{2}, 3\frac{1}{2}, 3\frac{1}{2}, 1\frac{1}{2})$ and $\text{Lmin}(v) = (4\frac{1}{2}, 2\frac{1}{2}, 2\frac{1}{2}, 2\frac{1}{2})$ while both Lmax and Lmin satisfy IIC, SYM and CON on the class of (balanced and therefore also) large core games. Hence, there are at least two different solutions on the class of large core games that satisfy all three axioms.

In order to get a characterization, CON is somewhat strengthened. Let ψ be a solution on the class of large core games. The solution ψ is said to satisfy AMON (aggregate monotonicity, see e.g. Young (1985)) if for every pair of large core games v and w with $w \in U(v)$ we have $\psi(v) \leq \psi(w)$.

It is not difficult to see that no egalitarian solution on the class of balanced games satisfies this condition. Restricted to the class of large core games, Lmax is an aggregate monotonic solution though. And since we already know that ϕ is equal to Lmax it is even quite straightforward to show

Theorem 9. *The solution Lmax satisfies IIC, SYM and AMON on the class of large core games.*

Proof. Firstly, since $\text{Lmax}(v)$ is defined as the maximal point in $C(v)$ according to the preference relation $>_{lex}$ on $\mathbb{R}^N$ it is clear that Lmax satisfies IIC. Secondly, since Lmax equals ϕ it is clear from the procedure that Lmax satisfies AMON. Finally, if two players i and j are symmetric in a game v it is obvious that i is a member of $P_k(v)$ if and only if j is a member of this coalition. Therefore they get the same amount in $\phi(v) = \text{Lmax}(v)$ □.

Finally we will show in this section that Lmax is the only solution on the class of large core games that satisfies these three axioms. To see this, let ψ be a core solution on the class of large core games that satisfies SYM, IIC and AMON.

In the proof the following construction is used. Let v be a (large core) game and let $T \neq N$ be a coalition. For each player $k \in T$, define

$$m(k) := \max\{v(S) - v(S \backslash k) | k \in S \subset T\}.$$

Now choose a number $K > 0$ such that $-K|S| < v(S)$ for all coalitions S and

$$\sum_{k \in T} m(k) - K|N \backslash T| < v(N).$$

Notice that this is viable since $N \backslash T$ is not the empty coalition by assumption. Define the game v^T by

$$v^T(S) := \begin{cases} v(S) & \text{if } S \subset T \text{ or } S = N \\ -K|S| & \text{else.} \end{cases}$$

Since the core of v is a subset of the core of v^T it is clear that v^T is balanced. Lemma 14 shows that it is even a large core game. Now we can show

Theorem 10. *The core solution ψ is equal to the solution ϕ.*

Proof. Let w be a large core game. We will show the equality of $\phi(v)$ and $\psi(v)$ by induction to the number of steps required to calculate $\phi(v)$.

A. Suppose that $P_1(v) = N$. Take $T := \emptyset$ and construct the large core game v^T. Then from the definition of v^T it is clear that all players are symmetric. So by SYM we know that

$$\psi(v^T) = \left(\frac{v^T(N)}{|N|}, \ldots, \frac{v^T(N)}{|N|} \right) = \left(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|} \right).$$

On the other hand, since $P_1(v) = N$, we know that $\phi(v)$ is also equal to the equal share point, so $\phi(v) = \psi(v^T)$. Moreover, $\psi(v^T) = \phi(v)$ must be an element of $C(v)$ by Lemma 7. Hence, since $C(v) \subset C(v^T)$, we get that $\psi(v) = \psi(v^T) = \phi(v)$ by IIC.

B. Suppose we already know that $\psi(w) = \phi(w)$ for all large core games w with $P_k(w) = N$. Take a large core game v with $P_k(v) \neq N$ and $P_{k+1}(v) = N$. We will show that $\psi(v) = \phi(v)$. To this end, note that the coalition $T := P_k(v)$ is not equal to the grand coalition by assumption, so we can construct the large core game v^T. Then we can also define the large core game $w \in U(v^T)^2$ by

$$w(S) := \begin{cases} v^T(S) & \text{if } S \neq N \\ \phi(v)(T) + |N \backslash T| \eta_k(v) & \text{if } S = N. \end{cases}$$

Since $w(S) = v^T(S) = v(S)$ if $S \subset T = P_k(v)$ and $w(S) = v^T(S) < v(S)$ for all other coalitions S unequal to N, we clearly have

$$\phi(w)_i = \begin{cases} \phi(v)_i & \text{if } i \in P_k(v) \\ \eta_k(v) & \text{if } i \not\in P_k(v) \end{cases}$$

by the particular choice of $w(N)$. It is also clear that $P_k(w) = N$, so $\psi(w) = \phi(w)$ by the induction hypothesis.

Next we will show that $\psi(v^T) = \phi(v)$, i.e.

$$\psi(v^T)_i = \begin{cases} \phi(v)_i & \text{if } i \in P_k(v) \\ \eta_{k+1}(v) & \text{if } i \not\in P_k(v). \end{cases}$$

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[2]The inequality $w(N) \geq v^T(N)$ follows from the definitions of $v^T(N)$ and $\eta_k(v)$.

To this end, let j be a player in $P_k(v)$. Then there is a coalition S with $j \in S$ and $S \subset P_k(v)$ such that $v(S) = \phi(v)(S)$. Then we also have

$$\psi(w)(S) = \phi(v)(S) = v(S) = v^T(S).$$

Further, by AMON, we know that $\psi(v^T) \leq \psi(w)$. So, since ψ is a core solution, it follows that $\psi(v^T)_i = \psi(w)_i$ for all $i \in S$, otherwise we would have

$$\psi(v^T)(S) < \psi(w)(S) = v^T(S).$$

Hence, in particular, $\psi(v^T)_j = \psi(w)_j$.

On the other hand, the players outside $P_k(v)$ get the same in $\psi(v^T)$ by SYM. So, they have to divide the amount

$$v^T(N) - \psi(v^T)(T) = v(N) - \psi(w)(T) = v(N) - \phi(v)(T)$$

equally among each other. However, since $T = P_k(v)$ and $P_{k+1}(v) = N$ this amount must be equal to $\eta_{k+1}(v)$. Hence, $\psi(v^T) = \phi(v)$.

So, now we have $\psi(v^T) = \phi(v) \in C(v)$. Hence, since $C(v) \subset C(v^T)$, we get that $\psi(v) = \psi(v^T)$ by IIC and $\psi(v) = \psi(v^T) = \phi(v)$. $\square$

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Appendix A. Proofs for Section 2

The aim of this appendix is to provide a proof of Theorem 2 as well as the background results used in that proof. First we give some general statements concerning convex games. Then we will give the proofs that concern the specific setup in Section 2.

It is a well-known fact that the space of convex games is a polyhedral cone, since it is defined by the inequalities

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T).$$

It is not so hard to show that this cone is of full dimension. Then a basic result in linear algebra states that a game v is an element of the interior of this cone if and only if for every convex game w there is a number $\varepsilon > 0$ such that $(1+\varepsilon)v - \varepsilon w$ is still convex. We will use this fact freely throughout this section. Further notice that

$$v(S) + v(T) = v(S \cap T) + v(S \cup T)$$

whenever S is a subset of T or T is a subset of S. So, the interior of the cone of convex games is given by those strict inequalities

$$v(S) + v(T) < v(S \cap T) + v(S \cup T)$$

for which S is not a subset of T and T is not a subset of S. A game that satisfies these strict inequalities is called *strictly* convex. Hence, a game v is an element of the interior of the cone of convex games if and only if it is strictly convex.

Lemma 11. *Let v be a strictly convex game. Let further x be an element of the core of v. Then the collection of coalitions that are tight on x is a chain.*

Proof. Suppose that S and T are two coalitions such that $x(S)=v(S)$ and $x(T)=v(T)$. Suppose further that S is not a subset of T and T is not a subset of S. Then, since v is an element of the interior of the cone of convex games and x is a core allocation, we would get

$$\begin{aligned} x(S\cap T)+x(S\cup T) &= x(S)+x(T) \\ &= v(S)+v(T) \\ &< v(S\cap T)+v(S\cup T) \\ &\le x(S\cap T)+x(S\cup T) \end{aligned}$$

Contradiction. Finally note that the coalitions $\emptyset$ and N are tight on x by definition. □

Given a convex game v and given a chain $\mathscr{S}$, it is our aim to construct a convex game v^* such that

$v^*\le v$,
$v^*(S)=v(S)$ whenever $S\in\mathscr{S}$, and
for any two consecutive coalitions S_m, $S_{m+1}\in\mathscr{S}$ the players in $S_{m+1}\backslash S_m$ are symmetric.

To this end, we introduce some notation. For any $S\subseteq N$, we denote the largest coalition in $\mathscr{S}$ that is a subset of S by $\overline{S}$, and we define $M:=\max\{v(S)-v(T)|S,T\subseteq N\}$.

Now, define the game v^* by

$$v^*(S)=v(\overline{S})+M(|\overline{S}|-|S|).$$

It is trivial that we have indeed $v^*\le v$, and that the players in $S_{m+1}\backslash S_m$ are symmetric for every pair of consecutive coalitions S_{m+1}, $S_m\in\mathscr{S}$. It remains to show that v^* is convex.

Lemma 12. *The game v^* is convex.*

Proof. Take two coalitions S and T. We have to show that

$$v^*(S)+v^*(T)\le v^*(S\cap T)+v^*(S\cup T).$$

Note that $\overline{S\cap T}=\overline{S}\cap\overline{T}$. The inclusion $\overline{S\cap T}\supseteq\overline{S}\cap\overline{T}$ follows directly from the facts that $\overline{S}\cap\overline{T}$ is a member of $\mathscr{S}$ and also a subset of $S\cap T$. The inclusion $\overline{S\cap T}\subseteq\overline{S}\cap\overline{T}$ follows directly from the fact that $\overline{S\cap T}$ is a subset of both $\overline{S}$ and $\overline{T}$.

Also note that $\overline{S \cup T} \supseteq \bar{S} \cup \bar{T}$, which follows from that fact both $\bar{S}$ and $\bar{T}$ are subsets of $\overline{S \cup T}$. Equality need not hold here.

To prove the lemma we distinguish the cases $\overline{S \cup T} \supset \bar{S} \cup \bar{T}$ and $\overline{S \cup T} = \bar{S} \cup \bar{T}$.

A. $\overline{S \cup T} = \bar{S} \cup \bar{T}$. Then

$$\begin{aligned} v^*(S) + v^*(T) &= v(\bar{S}) + v(\bar{T}) + M(|\bar{S}| - |S| + |\bar{T}| - |T|) \\ &\leq v(\bar{S} \cap \bar{T}) + v(\bar{S} \cup \bar{T}) \\ &= \; + M(|\bar{S} \cap \bar{T}| - |S \cap T| + |\bar{S} \cup \bar{T}| - |S \cup T|) \\ &= v(\overline{S \cap T}) + v(\overline{S \cup T}) \\ &= \; + M(|\overline{S \cap T}| - |S \cap T| + |\overline{S \cup T}| - |S \cup T|) \\ &= v^*(S \cap T) + v^*(S \cup T). \end{aligned}$$

B. $\overline{S \cup T} \supset \bar{S} \cup \bar{T}$. In this case the inequality

$$|\bar{S}| - |S| + |\bar{T}| - |T| \leq |\overline{S \cap T}| - |S \cap T| + |\overline{S \cup T}| - |S \cup T|$$

is strict, and by the choice of M it follows that

$$M(|\bar{S}| - |S| + |\bar{T}| - |T|) + v(\overline{S \cup T}) - v(\bar{S} \cup \bar{T}) \leq M(|\overline{S \cap T}| - |S \cap T| + |\overline{S \cup T}| - |S \cup T|).$$

This yields $v^*(S) + v^*(T) \leq v^*(S \cap T) + v^*(S \cup T)$ $\square$.

Lemma 13. *Let v be a strictly convex game. Then $\phi(v) = \phi(v^*)$.*

Proof. Define, for $0 \leq \lambda \leq 1$, the game $v(\lambda)$ by

$$v(\lambda) := (1 - \lambda)v + \lambda v^*.$$

Since ϕ satisfies CON, the maximal number λ^* for which $\phi(v)$ is equal to $\phi(v(\lambda))$ whenever λ is less or equal to λ^* exists.

It suffices to show that λ^* equals 1. So, assume that $\lambda^* < 1$. Then for $\varepsilon > 0$ (and $\varepsilon < 1 - \lambda^*$) we can define the game $w(\varepsilon)$ by

$$w(\varepsilon)(S) := \max\{v(\lambda^*)(S), v(\lambda^* + \varepsilon)(S)\}.$$

A. We will show that $\phi(v(\lambda^*))$ is an element of $C(w(\varepsilon))$. To this end, notice that $\phi(v)$ is an element of the core of both v and v^*. Then it is easy to check that $\phi(v)$ is an element of the core of both $v(\lambda^*)$ and $v(\lambda^* + \varepsilon)$. Hence, $\phi(v(\lambda^*)) = \phi(v)$ must also be an element of $w(\varepsilon)$.

B. We will show that $\phi(v(\lambda^* + \varepsilon))$ is also an element of $C(w(\varepsilon))$ for sufficiently small ε. To this end note that $\phi(v(\lambda^* + \varepsilon))$ is an element of $C(v(\lambda^* + \varepsilon))$ by definition. So we only need to show that $\phi(v(\lambda^* + \varepsilon))$ is an element of $C(v(\lambda^*))$ for sufficiently small ε. So, take a coalition S. To show:

$$\phi(v(\lambda^* + \varepsilon))(S) \geq v(\lambda^*)(S).$$

First, suppose that S is tight on $\phi(v(\lambda^*))$ in the game $v(\lambda^*)$. Now, since $\phi(v(\lambda^*)) =$

$\phi(v)$ is an element of both $C(v)$ and $C(v^*)$, and since $S_1, \ldots, S_k$ are the only tight coalitions on $\phi(v)$ in $C(v)$, it is clear that S is an element of $\mathscr{S}$, say $S = S_l$. Then

$$\begin{aligned}\phi(v(\lambda^* + \varepsilon))(S_l) &\geq v(\lambda^* + \varepsilon)(S_l)\\ &= ((1 - \lambda^* - \varepsilon)v(S_l) + (\lambda^* + \varepsilon)v^*(S_l)\\ &= ((1 - \lambda^* - \varepsilon)v^*(S_l) + (\lambda^* + \varepsilon)v^*(S_l) = v^*(S_l).\end{aligned}$$

Secondly, if is S is not tight on $\phi(v(\lambda^*))$ in the game $v(\lambda^*)$. Then

$$\phi(v(\lambda^*))(S) > v(\lambda^*)(S)$$

and since ϕ satisfies CON, it is clear that

$$\phi(v(\lambda^* + \varepsilon))(S) > v(\lambda^*)(S)$$

must still hold for sufficiently small ε. These two arguments together show that $\phi(v(\lambda^* + \varepsilon))$ is indeed an element of the core of $v(\lambda^*)$, and hence of the core of $w(\varepsilon)$, for sufficiently small ε.

C. Now recall that v is strictly convex. So, by the assumption that $\lambda^* < 1$ this is also true for $v(\lambda^*)$ and $v(\lambda^* + \varepsilon)$ for sufficiently small ε. Then it is easy to see that $w(\varepsilon)$ is convex for sufficiently small ε. In particular, $w(\varepsilon)$ is an element of the domain of ϕ.

Further, notice that the core of $w(\varepsilon)$ is a subset of both the core of $v(\lambda^*)$ and the core of $v(\lambda^* + \varepsilon)$. So, since we already saw in A and B that $\phi(v(\lambda^*))$ and $\phi(v(\lambda^* + \varepsilon))$ are both elements of the core of $w(\varepsilon)$ for sufficiently small ε, by IIC we get that

$$\phi(v) = \phi(v(\lambda^*)) = \phi(w(\varepsilon)) = \phi(v(\lambda^* + \varepsilon))$$

for sufficiently small ε. This contradicts the definition of λ^*. Hence, $\lambda^* = 1$ and $\phi(v) = \phi(v^*)$. □

Now we have gathered enough equipment to proceed with the goal of this Appendix. We will give a proof of

Theorem 1. *Let v be a convex game. Then $\phi(v)$ is a partial marginal of v.*

Proof. The proof is split up in two consecutive cases.

A. First, let v be a strictly convex game. Let $\mathscr{S}$ be the chain $\emptyset = S_1 \subset \cdots \subset S_k = N$ of all tight coalitions on $\phi(v)$. We will show that $\phi(v)$ equals the partial marginal $p(\mathscr{S}, v)$.

To this end, let j be a fixed player, and let m be the unique index such that j is a member of the coalition $X_{m+1} := S_{m+1} \backslash S_m$. Construct the game v^* as described at the start of this section. By Lemma 13 it suffices to show that

$$\phi(v^*)_j = \frac{v(S_{m+1}) - v(S_m)}{|S_{m+1} \backslash S_m|}.$$

Recall that $v(S_l) = v^*(S_l)$ for all coalitions in the chain $\mathscr{S}$. So, by Lemma 13 we get

$$\phi(v^*)(S_l) = \phi(v)(S_l) = v(S_l) = v^*(S_l)$$

for every coalition S_l in the chain. In particular we get

$$\begin{aligned}\phi(v^*)(S_{m+1}\backslash S_m) &= \phi(v^*)(S_{m+1}) - \phi(v^*)(S_m) \\ &= v^*(S_{m+1}) - v^*(S_m) = v(S_{m+1}) - v(S_m).\end{aligned}$$

Finally, notice that all players in $S_{m+1}\backslash S_m$ are symmetric in v^*. Hence, by SYM and the previous equality we get the above formula.

B. Now let v be an arbitrary convex game. Then we can take a sequence $(v^k)_{k\in\mathbb{N}}$ strictly convex convex games that converges to v. We have just shown that each $\phi(v^k)$ is a partial marginal. Then, using the fact that ϕ satisfies CON, it is easy to see that $\phi(v)$ must also be a partial marginal. □

Appendix B. Proof for Section 3

Lemma 14. *The game v^T is a large core game.*

Proof. Suppose that v^T is not a large core game. Then there is an allocation x such that $x(S) \geq v^T(S)$ for all coalitions $S \subset N$ and, moreover, there is no core allocation y in $C(v^T)$ with $y \leq x$. Now notice that we can choose this allocation x even such that for every player i there is a coalition S_i with $i \in S_i$ and $x(S_i) = v^T(S_i)$[3]. We will derive a contradiction in four steps.

A. First we will show for every coalition S_i that either S_i is a subset of T or S_i has an empty intersection with T. Suppose that this is not so. Then there is a coalition S_i for which $S_i \cap T$ and $S_i\backslash T$ are both not empty. Then we can calculate that

$$\begin{aligned}-K|S_i| = v^T(S_i) = x(S_i) &= x(S_i \cap T) + x(S_i\backslash T) \\ &\geq v^T(S_i \cap T) + x(S_i\backslash T) \\ &= v(S_i \cap T) + x(S_i\backslash T) \\ &> -K|S_i \cap T| + x(S_i\backslash T) \\ &\geq -K|S_i \cap T| - K|S_i\backslash T| = -K|S_i|.\end{aligned}$$

Contradiction.

B. Now take a player k in T. We will show that $x_k \leq m(k)$. To this end, notice that S_k must be a subset of T by A, since k is a member of both S_k and T. So,

$$\begin{aligned}v(S_k) = v^T(S_k) = x(S_k) &= x_k + x(S_k\backslash k) \\ &\geq x_k + v^T(S_k\backslash k) \\ &= x_k + v(S_k\backslash k).\end{aligned}$$

[3] If x does not yet satisfy this condition, we can construct in n iterations an allocation that does by lowering in step k the payoff to player k in the current allocation x^{k-1} by the non-negative amount $\min\{x^{k-1}(S) - v^T(S) | S \ni k\}$. The final allocation x^n of this procedure is not an element of the core of v^T by the way the initial allocation x is chosen.

Hence, $x_k \leq v(S_k) - v(S_k \backslash k) \leq m(k)$.

C. Now let k be a player not in T. We will show that $x_k = -K$. To this end, notice that $x_j \geq -K$ for every player j since x is a core element. Further, since $k \in S_k$ implies that S_k is not a subset of T, we know that

$$x(S_k) = v^T(S_k) = -K|S_k|$$

by the definition of v^T. These two observations together imply that $x_j = -K$ for every player in j in S_k. In particular, $x_k = -K$.

D. Finally, using B and C, we get

$$x(N) = x(T) + x(N \backslash T) \leq \sum_{k \in T} m(k) - K|N \backslash T| < v^T(N)$$

where the strict inequality follows from the choice of K. However, the inequality $x(N) < v^T(N)$ contradicts the choice of x. □

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